Admissibility for multi-conclusion consequence relations and universal classes

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plan

- single-conclusion consequence relations and quasivarieties
- multi-conclusion consequence relations and universal classes
- application in intuitionistic/modal logic

φ, ψ - formulas
Γ, Δ - finite sets of formulas

Γ/φ - (sinlge-conclusion) rule

$\vdash$ - single-conclusion consequence relation (scr): a relation $\vdash$ s.t.

- $\varphi \vdash \varphi$
- if $\Gamma \vdash \varphi$, then $\Gamma, \Delta \vdash \varphi$
- if $\Gamma \vdash \psi$ for all $\psi \in \Delta$ and $\Delta \vdash \varphi$, then $\Gamma \vdash \varphi$
- if $\Gamma \vdash \varphi$, then $\sigma(\Gamma) \vdash \sigma(\varphi)$

$\mathrm{Th}(\vdash) = \{\varphi \in \mathit{Formulas} \mid \vdash \varphi\}$ - theorems of $\vdash$

quasivarieties

quasi-identities look like

$$(\forall \bar{x})\ s_1(\bar{x}) \approx t_1(\bar{x}) \wedge \cdots \wedge s_n(\bar{x}) \approx t_n(\bar{x}) \ \rightarrow\ s(\bar{x}) \approx t(\bar{x})$$

quasivarieties look like Mod(quasi-identities)

These are classes closed under subalgebras, products and ultraproducts

$\mathsf{SPP_U}(\mathcal{K})$ - a least quasivariety containing $\mathcal{K}$

correspondence

scr $\vdash$	$\leftrightsquigarrow$	quasivariety $\mathcal{Q}$
logical connectives	$\leftrightsquigarrow$	basic operations
theorems	$\leftrightsquigarrow$	valid identities
single-conclusion der. rules	$\leftrightsquigarrow$	valid quasi-identities
Th($\vdash$)	$\leftrightsquigarrow$	free algebra

admissibility for scrs

$\vdash_r$ - a least scr containing the rule r and extending $\vdash$

r is <u>admissible for</u> $\vdash$ if $\mathrm{Th}(\vdash) = \mathrm{Th}(\vdash_r)$

$\vdash$ is <u>structurally complete</u> if every single-conclusion admissible rule is derivable

Theorem (folklore)

Γ/φ is admissible for $\vdash$ iff

$$(\,\forall\gamma \in \Gamma,\ \vdash \sigma(\gamma)\,) \quad \text{yields} \quad \vdash \sigma(\varphi)$$

for every substitution σ

admissibility for quasivarieties

q - quasi-identity, $\mathcal{Q}$ - quasi-variety

q is admissible for $\mathcal{Q}$ if $\mathcal{Q}$ and $\mathcal{Q} \cap \mathrm{Mod}(q)$ satisfy the same identities

$\mathcal{U}$ is structurally complete is if every admissible for $\mathcal{U}$ quasi-identity holds in $\mathcal{U}$.

admissibility for quasivarieties

$\mathcal{Q}$ - quasivariety, $\mathbf{A}$ - algebra, $\mathrm{Con}(\mathbf{A})$ - congruences of $\mathbf{A}$
$\mathrm{Con}_{\mathcal{Q}}(\mathbf{A}) = \{\alpha \in \mathrm{Con}(\mathbf{A} \mid \mathbf{A}/\alpha \in \mathcal{Q}\}$

Fact [Bergman]

$\mathrm{Con}_{\mathcal{Q}}(\mathbf{A})$ has a least congruence $\rho_{\mathbf{A}}$.

$\mathbf{T}$ -algebra of terms over a denumerable set of variables

$\mathbf{F} = \mathbf{T}/\rho_{\mathbf{T}}$ - free algebra for $\mathcal{Q}$

Theorem (Bergman)

q is admissible for $\mathcal{Q}$ iff $\mathbf{F} \models q$.

mcrs

$\Gamma, \Gamma', \Delta, \Delta'$ - finite sets of formulas

Γ/Δ - (multi-conclusion) rule

$\vdash$ - multi-conclusion consequence relation (mcr): a relation $\vdash$ s.t.

- $\varphi \vdash \varphi$;
- if $\Gamma \vdash \Delta$, then $\Gamma, \Gamma' \vdash \Delta, \Delta'$;
- if $\Gamma \vdash \Delta, \varphi$ and $\Gamma, \varphi \vdash \Delta$, then $\Gamma \vdash \Delta$;
- if $\Gamma \vdash \Delta$, then $\sigma(\Gamma) \vdash \sigma(\Delta)$.

$\mathrm{Th}(\vdash) = \{\varphi \in \mathit{Formulas} \mid \vdash \varphi\}$ - theorems
$\mathrm{mTh}(\vdash) = \{\Delta \subseteq_{fin} \mathit{Formulas} \mid \vdash \Delta\}$ - multi-theorems

universal classes

basic universal sentences look like

$$(\forall \bar{x})\ s_1(\bar{x}) \approx t_1(\bar{x}) \wedge \cdots \wedge s_n(\bar{x}) \approx t_n(\bar{x}) \rightarrow s_1'(\bar{x}) \approx t_1'(\bar{x}) \vee \cdots \vee s_n'(\bar{x}) \approx t_n'(\bar{x})$$

universal classes look like

Mod(basic universal sentences)

These are classes closed under subalgebras and elementary equivalence

$\mathsf{SP_U}(\mathcal{K})$ - a least universal class containing $\mathcal{K}$

correspondence

mcr $\vdash$	$\leftrightsquigarrow$	universal class $\mathcal{U}$
logical connectives	$\leftrightsquigarrow$	basic operations
theorems	$\leftrightsquigarrow$	valid identities
multi-theorems	$\leftrightsquigarrow$	valid multi-identities
derivable rules	$\leftrightsquigarrow$	valid basic universal sentences
single-conclusion der. rules	$\leftrightsquigarrow$	valid quasi-identities
$\mathrm{Th}(\vdash)$	$\leftrightsquigarrow$	free algebra
$\mathrm{mTh}(\vdash)$	$\leftrightsquigarrow$	???

admissibility for mcr

$r = \Gamma/\delta$ - single conclusion rule
$\vdash_r$ - least mcr containing the rule r and extending $\vdash$

r is admissible for $\vdash$ if $\mathrm{mTh}(\vdash) = \mathrm{mTh}(\vdash_r)$

r is weakly admissible for $\vdash$ if $\mathrm{Th}(\vdash) = \mathrm{Th}(\vdash_r)$

r is narrowly admissible for $\vdash$ if for every substitution σ

$$(\forall \gamma \in \Gamma \vdash \sigma(\gamma)) \ \textit{yields} \ \vdash \sigma(\delta)$$

Theorem (Iemhoff)

Γ/δ is admissible for $\vdash$ iff for every substitution σ and every finite set of furmulas Σ

$$(\ \forall \gamma \in \Gamma, \vdash \sigma(\gamma), \Sigma\ ) \quad \text{yields} \quad \vdash \sigma(\delta), \Sigma$$

structural completeness for mcrs

$\vdash$ is (strongly, widely) structurally complete if every (weakly, narrowly) admissible for $\vdash$‘ single-conclusion rule belongs to $\vdash$ ‘

admissibility for universal classes

$q = (\forall \bar{x})\ s_1(\bar{x}) \approx t_1(\bar{x}) \wedge \cdots \wedge s_n(\bar{x}) \approx t_n(\bar{x}) \to s(\bar{x}) \approx t(\bar{x})$
a q-identity,
$\mathcal{U}$ - universal class

q is admissible for $\mathcal{U}$ if $\mathcal{U}$ and $\mathcal{U} \cap \mathrm{Mod}(q)$ satisfy the same muti-identities (positive basic universal sentences)

q is weakly admissible for $\mathcal{U}$ if $\mathcal{U}$ and $\mathcal{U} \cap \mathrm{Mod}(q)$ satisfy the same identities

q is narrowly admissible for $\mathcal{U}$ if for every substitution σ

$$(\ \forall i \leqslant n,\ \mathcal{U} \models \sigma(s_i) \approx \sigma(t_i)\ ) \quad \text{yields} \quad \mathcal{U} \models \sigma(s) \approx \sigma(t)$$

$\mathcal{U}$ is (stongly, widely) structurally complete if every (weakly, narrowly) admissible for $\mathcal{U}$ quasi-identity is valid in $\mathcal{U}$

$\mathcal{U}$ - universal class, **A** - algebra, Con(**A**) - congruences of **A**
$\mathrm{Con}_{\mathcal{U}}(\mathbf{A}) = \{\alpha \in \mathrm{Con}(\mathbf{A} \mid \mathbf{A}/\alpha \in \mathcal{U}\}$

$\mathrm{Con}_{\mathcal{U}}^{min}(\mathbf{A})$ - the set of minimal congruences in $\mathrm{Con}_{\mathcal{U}}(\mathbf{A})$

Key Fact

For every $\alpha \in \mathrm{Con}_{\mathcal{U}}(\mathbf{A})$ there exists $\gamma \in \mathrm{Con}_{\mathcal{U}}^{min}(\mathbf{A})$ s.t

$$\gamma \subseteq \alpha$$

Define

$\mathcal{F}_{\mathcal{U}} = \{\mathbf{T}/\gamma \mid \gamma \in \mathrm{Con}_{\mathcal{U}}^{min}(\mathbf{T})\}$ - free family for $\mathcal{U}$

(**T** - an algebra of terms)

characterization

$\mathcal{U}$ - universal class
F - free algebra (of denumerable rank) for $\mathsf{SP}(\mathcal{U})$
$\mathcal{F}$ - free family for $\mathcal{U}$
q - quasi-identity

Theorem

- q is admissible for $\mathcal{U}$ iff $\mathcal{F} \models q$
- q is weakly admissible for $\mathcal{U}$ iff $\mathbf{F} \in \mathsf{SP}(\mathcal{U} \cap \mathsf{Mod}(q))$
- q is narrowly admissible for $\mathcal{U}$ iff $\mathbf{F} \models q$

Corollary

- $\mathcal{U}$ is structurally complete iff $\mathsf{SP}(\mathcal{U}) = \mathsf{SPP}_{\mathsf{U}}(\mathcal{F}_{\mathcal{U}})$
- $\mathcal{U}$ is strongly structurally complete iff $\mathbf{F} \in \mathsf{SP}(\mathcal{U} \cap \mathcal{Q})$ yields $\mathcal{U} \subseteq \mathcal{Q}$ for every quasivariety $\mathcal{Q}$
- $\mathcal{U}$ is widely structurally complete iff $\mathsf{SPP}_{\mathsf{U}}(\mathbf{F}) = \mathsf{SP}(\mathcal{U})$.

dependence

wide stuructural completeness

$\Downarrow \quad \not\Uparrow$

strong structural completeness

$\Downarrow \quad \not\Uparrow$

structural completeness

an application

Blok-Esakia isomorphism

Theorem (Blok, Esakia, Jeřábek)

There is an isomorphism

$$\sigma : \mathrm{mExt}\,\mathbf{Int} \to \mathrm{mExt}\,\mathbf{Grz}.$$

Int - intuitionistic logic as a mcr
mExt **Int** - lattice of its extensions

Grz - modal Grzegorczyk logic as a mcr
mExt **Grz** - lattice of its extensions

closure algebras and Heyting algebras

closure algebras $=$ modal algebras satisfying $\Box\Box p = \Box p \leqslant p$

M - closure algebras
$\mathsf{O}(\mathbf{M}) = \{\Box \mathsf{p} \mid \mathsf{p} \in \mathsf{M}\}$ - Heyting algebras of open elements of **M**

Theorem (McKinsey, Tarski '46)

For a Heyting algebra **H** the exists a closure algebra $\mathsf{B}(\mathbf{H})$ s.t.

- $\mathsf{OB}(\mathbf{H}) = \mathbf{H}$;
- if $\mathbf{H} \leqslant \mathsf{O}(\mathbf{M})$, then $\mathsf{B}(\mathbf{H}) \cong \langle H \rangle_{\mathbf{M}}$

$\mathcal{W}$ - u. class of closure algebras, $\mathcal{U}$ - u. class of Heyting algebras

$\rho(\mathcal{W}) = \{\mathsf{O}(\mathbf{M}) \mid \mathbf{M} \in \mathcal{W}\}$ - universal class of Heyting algebras
$\sigma(\mathcal{U}) = \mathsf{SP_U}\{\mathsf{B}(\mathbf{H}) \mid \mathbf{H} \in \mathcal{U}\}$ - universal class of Grzegorczyk algebras

Blok-Esakia algebraically

There mappings

$$\rho\colon \mathsf{L}_{\mathsf{U}}(\mathcal{G}rz) \to \mathsf{L}_{\mathsf{U}}(\mathcal{H}ey)$$
$$\sigma\colon \mathsf{L}_{\mathsf{U}}(\mathcal{H}ey) \to \mathsf{L}_{\mathsf{U}}(\mathcal{G}rz)$$

are mutually inverse lattice isomorphisms

$\mathcal{H}ey$ - class of all Heyting algebras
$\mathsf{L}_{\mathsf{U}}(\mathcal{H}ey)$ - lattice of its universal subclasses

$\mathcal{G}rz$ - class of all Grzegorczyk algebras
$\mathsf{L}_{\mathsf{U}}(\mathcal{G}rz)$ - lattice of its universal subclasses

preservation

Theorem

$\mathcal{U}$ - universal class of Heyting algebras. Then $\mathcal{U}$ is (widely, strongly) structurally complete iff $\sigma(\mathcal{U})$ is (widely, strongly) structurally complete

Corollary

$\vdash$ - mcr extending **Int**. Then $\vdash$ is (widely, strongly) structurally complete iff $\sigma(\vdash)$ is (widely, strongly) structurally complete

The end

Thank you!